# The stability of a viscous fluid between rotating cylinders with an axial flow

By R. C. DIPRIMA
Department of Mathematics, Rensselaer Polytechnic Institute, Troy, New York

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The stability of a viscous fluid between two concentric rotating cylinders with an axial flow is investigated. It is assumed that the cylinders are rotating in the same direction and that the spacing between the cylinders is small. The critical Taylor number is computed for small Reynolds number associated with the axial flow. It is found that the critical Taylor number increases with increasing Reynolds number.

## 1. Introduction

The stability characteristics of viscous flow between concentric rotating cylinders with an axial flow are of interest in several technical areas including paper making (Appel 1959), and the design of rotating electrical machinery (Kaye & Elgar 1957).

Goldstein (1937) considered this problem theoretically for the case of the outer cylinder at rest and the gap between the cylinders small compared to the mean radius. He found that the critical Taylor number $T$ (associated with the angular velocity of the inner cylinder)* increases initially, as the Reynolds number $R$ (associated with the axial velocity) increases from 0 to a value of about 20, and then decreases quite rapidly as $R$ increases to about 25.

Cornish (1933) and Fage (1938) used pressure drop measurements to determine experimentally the critical angular velocity of the inner cylinder for given axial flows in the case in which the outer cylinder is at rest. The results of both of these investigations indicated that the critical Taylor number increases with increasing Reynolds number. However, the results of the two investigations are considerably different from one another. The results of Cornish (1933) give Taylor numbers larger by a factor of about three than those found by Fage (1938). Indeed an extrapolation of Cornish's results to the case of zero axial velocity gives a value of the critical Taylor number about three times larger than the known correct value. For this reason Goldstein (1937) concluded that Cornish was measuring a different phenomenon from that considered by himself. On the other hand, the results of Fage (1938), while giving the correct Taylor number when extrapolated to the case of zero axial velocity, show only a very slight increase of the critical Taylor number with increasing Reynolds number until fairly large values of $R$. More recently Kaye & Elgar (1957) have considered

* See equations (5) for a precise definition of the Taylor number and the Reynolds number.

this problem experimentally using an apparatus which allowed visual and photographic as well as hot-wire measurements. Their measurements (again, for the case of the outer cylinder at rest) show that the critical Taylor number increases fairly rapidly with increasing Reynolds number. Their results are in disagreement with the results of Cornish (1933) and Fage (1938).

This paper is concerned with a theoretical analysis of the stability of a viscous fluid between concentric rotating cylinders with a small axial flow. The characteristic value problem is formulated in § 2 using the assumption that the spacing between the cylinders is small compared to the mean radius. In the case in which the cylinders are rotating in the same direction, the characteristic value problem is solved in § 3 with the axial velocity replaced by its average value. Several methods of solving this problem are discussed. In § 4 the same problem is treated using a parabolic axial velocity distribution and the results are compared with those of § 3. It is found in both cases that the critical Taylor number increases rapidly (more rapidly for the latter case) from the correct value ($T \sim 1710$) at $R = 0$ to a value of abut 7000 at $R = 60$. The results further show that the frequency and wavelength of the distribution are sensitive to the approximation used for the axial velocity. In both cases the results of the present analysis are in complete disagreement with the work of Goldstein (1937), Cornish (1933) and Fage (1938) and in qualitative agreement with the work of Kaye & Elgar (1957).

## 2. The characteristic value problem

Consider two infinitely long concentric cylinders. Let $(r, \theta, z)$ be cylindrical coordinates, and let $R_1, R_2, \Omega_1$ and $\Omega_2$ denote the radii and angular velocities of the inner and outer cylinders, respectively. If $u_r, u_\theta, u_z$ denote the components of velocity in the increasing $r$, $\theta$, and $z$ direction and $p$ denotes the pressure, the Navier–Stokes equations admit a steady solution of the form

$$u_r = 0, \quad u_\theta = V(r), \quad u_z = W(r), \quad \partial p/\partial z = \text{constant}. \tag{1}$$

Now superimpose on this steady motion a rotationally symmetric disturbance of a form such that the $\theta$ component of velocity is

$$u_\theta(r, z, t) = V(r) + v(r)\, e^{i(\sigma t + \lambda z)}. \tag{2}$$

In general $\sigma$ will be complex. The motion will be stable or unstable as the imaginary part of $\sigma$ is positive or negative, respectively. We shall be concerned in this paper with the case of neutral stability for which the imaginary part of $\sigma$ is equal to zero. Notice this is different from the case for zero axial flow where the instability is of a stationary cellular nature and $\sigma$ is set identically equal to zero. Substituting for $u_r, u_\theta, u_z$ and $p$ in the Navier–Stokes equations and neglecting quadratic terms in the disturbance velocities leads to a sixth-order system of linear homogeneous differential equations. The requirement of no slip at the boundaries gives six homogeneous boundary conditions.

In the case that the distance between the cylinders, $d = R_2 - R_1$, is small compared to the mean radius, $R_0 = \frac{1}{2}(R_1 + R_2)$, this system of equations can be considerably simplified. Neglecting terms of order $d/R_0$, the angular velocity $\Omega = V/r$

and the axial velocity $W$ can be approximated by linear and parabolic profiles. To the same order of accuracy the non-dimensional disturbance equations for neutral stability may be written as

$$(D^2-a^2)^2 u - i\{\beta - aRf(x)\}(D^2-a^2)u + 12iaRu = -a^2Tg(x)v, \tag{3}$$

$$(D^2-a^2)v - i\{\beta - aRf(x)\}v = u, \tag{4}$$

where

$$\left.\begin{aligned} r = R_0 + dx, \quad D = d/dx, \quad a = \lambda d, \quad \beta = \sigma_r d^2/\nu, \\ W(x) = W_{av}f(x), \quad f(x) = 6(\tfrac{1}{4}-x^2), \quad \Omega(x) = \Omega_{av}g(x), \\ g(x) = 1 - 2\frac{1-k}{1+k}x, \quad k = \Omega_2/\Omega_1, \quad k \neq -1, \\ R = |W_{av}|\,d/\nu, \quad T = -4A\Omega_{av}d^4/\nu^2, \quad A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}. \end{aligned}\right\} \tag{5}$$

$\sigma_r$ denotes the real part of $\sigma, u$ has been redefined as $(2Ad^2/\nu)u$, and $W_{av}$ and $\Omega_{av}$ denote the average axial and angular velocities, respectively. The parameter $T$ associated with the angular velocity is commonly referred to as the Taylor number, and the parameter $R$ associated with the axial velocity will be referred to as the Reynolds number. Equations (3) and (4) are identical with (25) and (27) given by Goldstein (1937); however the notation is different.

The boundary conditions are

$$u = v = Du = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}. \tag{6}$$

Notice that when there is no axial flow, $R = 0$, $\beta = 0$, (3) and (4) reduce to those for the classical Taylor problem as given by Chandrasekhar (1954); and when the cylinders are stationary, i.e. $T = 0$, the equations uncouple and (3) reduces to the Orr–Sommerfeld equation for the stability of a viscous flow between parallel plates. Thus for small values of $T$ and large $R$ we can expect an instability of the Tollmien–Schlichting type, and for small $R$ and large $T$ the instability will be of a non-stationary cellular nature. Kaye & Elgar (1957) have pointed out that there are four regions depending upon the values of $R$ and $T$. This is illustrated qualitatively in figure 1. In this paper we shall only consider the case in which $R$ is small. That is, we are interested in determining the lower left branch of the curves depicted in figure 1.

The system of equations (3) and (4) together with the boundary conditions (6) determine a non-self adjoint characteristic value problem for $T$ as a function of $R$, $a$, $\beta$ and $k$. Mathematically the problem is the following: for given real values of $R$ and $k$ we wish to find the minimum positive value of $T$ with respect to real positive values of $a$ and real values of $\beta$. This minimum value of $T$, $T_c$ say, determines the value of $\Omega_1$ at which a secondary motion will first occur; the corresponding values $a_c$ and $\beta_c$ give the wave number and the frequency of the secondary motion. The wave velocity $c = \sigma_r/\lambda$ can be conveniently expressed in dimensionless form by $c/W_{av} = \beta/aR$.

It is clear that the determination of $T_c$ is a rather complicated four-parameter problem and we shall now restrict ourselves to the case in which the cylinders rotate in the same direction. In this case it is known that, for $R = 0$, $\Omega(x)$ can be approximated by its average value (even for $\Omega_2 = 0$) with only a very small error in the determination of $T_c$ (see Chandrasekhar 1960). From the form of equation

(3) it can be anticipated that this will still be a satisfactory approximation for $R \geqslant 0$ as long as $k \geqslant 0$. Thus we replace $g(x)$ by its average value of unity in (3).

Even with this approximation the mathematical problem is still difficult because of the variable coefficients in (3) and (4) arising from the axial velocity. The possibility of approximating $W(x)$ by its average value is a much more

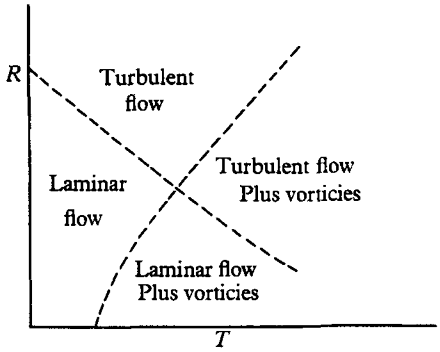


FIGURE 1. The four régimes of flow following Kaye & Elgar (1957).

delicate question than that of replacing $\Omega(x)$ by its average value for $k \geqslant 0$. Indeed from studies of the Orr–Sommerfeld equation we can expect that for large $R$ the characteristic value will be very sensitive to variations in $W(x)$. However, for the case of small $R$ considered here the situation should not be as serious, and in the next section we shall replace $W(x)$ by its average value in (3) and (4). With these approximations we obtain a characteristic value problem which can be solved accurately and fairly easily. In § 4 we shall treat by approximate methods the much more difficult problem where a parabolic profile is used for $W(x)$.

## 3. Case 1; W and Ω approximated by their average values

In this case we replace $f(x)$ and $g(x)$ each by unity and (3) and (4) reduce to

$$\{(D^2-a^2)^2 - i\xi(D^2-a^2) + 12iaR\}\,u = -Ta^2v, \tag{7}$$

$$\{(D^2-a^2) - i\xi\}\,v = u, \tag{8}$$

where we have let $\xi = \beta - aR$. The characteristic value problem defined by the above equations and the boundary conditions (6) can be treated to a high degree of accuracy by convenient approximate techniques such as the Galerkin method or it can be solved exactly by the techniques used by DiPrima (1960) to treat similar problems. In this section we shall compare several such methods.

To solve the problem exactly, let $Z = T^{\frac{1}{2}}$, redefine $u$ as $iaZu$ and define the functionals $\phi(u,v)$ and $\psi(u,v)$ by

$$\phi(u,v) = \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \{[(D^2-a^2)\,u]^2 + i\xi[(Du)^2+a^2u^2] + 12iaRu^2 - (Dv)^2 - a^2v^2 - i\xi v^2\}\,dx, \tag{9}$$

$$\psi(u,v) = \int_{-\frac{1}{2}}^{\frac{1}{2}} uv\,dx. \tag{10}$$

The above characteristic value problem is equivalent to the following variational principle: Among all functions $u$ and $v$ satisfying the boundary conditions (6), that set which makes the first variation of $I(u,v) = \phi(u,v) - iaZ\psi(u,v)$ vanish necessarily satisfy equations (7) and (8). The proof of this variational principle and the details of the solution are very similar to problems treated by DiPrima (1960) and hence only a brief summary of the method of solution will be given here.

To solve the variational problem we first note that the solution can be split into even and odd functions about $x = 0$. For the even solution we expand $u$ and $v$ in a complete set of even orthonormal functions on $-\frac{1}{2} \leqslant x \leqslant \frac{1}{2}$. Appropriate series are

$$u(x) = \sum_{n=1}^{\infty} a_n E_n(x), \quad v(x) = \sum_{n=1}^{\infty} b_n E_n(x), \tag{11}$$

where the $E_n(x) = 2^{\frac{1}{2}} \cos(2n-1)\pi x$. The boundary conditions $u = v = 0$ at $x = \pm\frac{1}{2}$ are automatically satisfied; the boundary conditions $Du = 0$ at $x = \pm\frac{1}{2}$ introduce the constraining condition

$$\Gamma = \sum_{n=1}^{\infty} (-1)^{n+1}(2n-1)a_n = 0. \tag{12}$$

Substituting the series for $u$ and $v$ in the expression for $I$ and making use of the orthonormal properties of the $E_n(x)$ gives $I$ as a function of the $a_n$ and $b_n$. The vanishing of the first variation in $I$ subject to the constraint (12) requires the vanishing of the partial derivatives of $I - \mu\Gamma$ with respect to the $a_n$ and $b_n$. Here $\mu$ is a Lagrange multiplier. This leads to two simultaneous linear non-homogeneous equations for $a_n$ and $b_n$. Solving for $a_n$ and substituting in the condition $\Gamma = 0$ gives the following equation for $T$ as a function of $R$, $a$, and $\xi$

$$\sum_{n=1}^{\infty} \frac{(2n-1)^2(A_n+i\xi)}{\{A_n(A_n+i\xi)+12iaR\}(A_n+i\xi)-a^2T} = 0, \tag{13}$$

where $A_n = (2n-1)^2\pi^2 + a^2$. This equation after separation into its real and imaginary parts and some simple algebraic manipulations to improve the convergence of one of the series (the convergence is like $(2n-1)^{-4}$) can be solved by trial and error methods. For given values of $R$ and $a$ there are a sequence of pairs of real numbers $(\xi, T)$ that satisfy the two equations. Choosing the smallest positive value of $T$ and the corresponding value of $\xi$ and then minimizing with respect to $a$ determines $T_c$ and the corresponding values of $a$ and $\xi$. In carrying out the numerical computations a sufficient number of terms in the series were used to insure that there would be less than 1 % error in the determination of $T_c$. The results of the computations for $R = 5{\cdot}17$, $20{\cdot}67$, $25{\cdot}84$ and 100 are given in table 1. The odd solution which gives a larger value of $T_c$ can be treated in a similar manner.

Although the above method can be used to determine the characteristic values as accurately as one wishes, satisfactory answers can be obtained by approximate methods that are easier to use. Also in solving equation (13) it is desirable to have first approximations to the roots as a starting point for the computations. Approximate solutions can be obtained conveniently by the Galerkin method. This method consists of expanding $u$ and $v$ in sets of complete functions, preferably orthogonal, that satisfy the boundary conditions and then requiring the error in the equations for $u$ and $v$ to be orthogonal to the expansion functions for $u$ and $v$. Thus for the even solution of equations (7) and (8) we write $u$ and $v$ as

$$u(x) = \sum_{n=1}^{M} a_n C_n(x), \quad v(x) = \sum_{n=1}^{M} b_n E_n(x), \tag{14}$$

| Investigation | $R$ | $a$ | $\beta$ | $\beta/aR$ | $T$ |
|---|---|---|---|---|---|
| 1 | 0 | 3·10 | 0 | — | 1,728 |
| 4 | 0 | 3·10 | 0 | — | 1,715 |
| 1 | 1 | 3·11 | 2·53 | 0·814 | 1,730 |
| 2 | 1 | 3·11 | 2·53 | 0·814 | 1,710 |
| 1 | 2 | 3·11 | 5·06 | 0·814 | 1,734 |
| 2 | 2 | 3·12 | 5·08 | 0·814 | 1,714 |
| 4 | 5 | 3·1 | 12·6 | 0·813 | 1,753 |
| 1 | 5·17 | 3·12 | 13·13 | 0·816 | 1,769 |
| 2 | 5·17 | 3·13 | 13·16 | 0·813 | 1,750 |
| 3 | 5·17 | 3·1+ | 13·02 | 0·812 | 1,748 |
| 1 | 10·34 | 3·16 | 26·63 | 0·815 | 1,890 |
| 2 | 10·34 | 3·17 | 26·68 | 0·814 | 1,870 |
| 4 | 20·0 | 3·4 | 55·7 | 0·819 | 2,309 |
| 1 | 20·67 | 3·4 | 57·64 | 0·820 | 2,360 |
| 2 | 20·67 | 3·4 | 57·52 | 0·818 | 2,338 |
| 3 | 20·67 | 3·4 | 57·38 | 0·816 | 2,340 |
| 1 | 25·84 | 3·55 | 75·56 | 0·824 | 2,698 |
| 3 | 25·84 | 3·55 | 75·33 | 0·821 | 2,678 |
| 1 | 40 | 4·2 | 141·2 | 0·840 | 3,893 |
| 2 | 40 | 4·23 | 142·0 | 0·839 | 3,860 |
| 4 | 40 | 4·2 | 140·9 | 0·839 | 3,881 |
| 1 | 60 | 5·15 | 267·7 | 0·866 | 5,988 |
| 2 | 60 | 5·15 | 267·1 | 0·864 | 5,925 |
| 4 | 60 | 5·2 | 270·2 | 0·866 | 5,962 |
| 1 | 80 | 5·90 | 417·9 | 0·885 | 8,368 |
| 2 | 80 | 5·92 | 418·7 | 0·884 | 8,261 |
| 4 | 80 | 6·0 | 425 | 0·885 | 8,319 |
| 1 | 100 | 6·54 | 588·2 | 0·899 | 10,960 |
| 2 | 100 | 6·58 | 591·4 | 0·899 | 10,800 |
| 3 | 100 | 6·55 | 587·2 | 0·896 | 10,800 |
| 4 | 100 | 6·6 | 594·0 | 0·900 | 10,876 |

TABLE 1. Critical Taylor numbers and corresponding values of $a$ and $\beta$ and $\beta/aR$ for assigned values of $R$. Investigation 1 refers to Galerkin method with $M = 1$; 2 refers to Galerkin method with $M = 2$; 3 is the exact solution; 4 refers to the results of Chandrasekhar (1960). ($W$ and $\Omega$ are approximated by their average values.)

where the $E_n(x)$ are the functions defined earlier, and the $C_n(x)$ are even orthonormal functions* satisfying $C_n$ and $DC_n = 0$ at $x = \pm\frac{1}{2}$. Substituting these series in equations (7) and (8), multiplying (7) by $C_j(x)$ and (8) by $E_j(x)$ for $j = 1, 2, \ldots, M$, and integrating from $-\frac{1}{2}$ to $+\frac{1}{2}$ gives $2M$ simultaneous

* The functions $C_n(x)$ are of the form $(\cosh \lambda_n x)/(\cosh \frac{1}{2}\lambda_n) - (\cos \lambda_n x)/(\cos \frac{1}{2}\lambda_n)$, where the $\lambda_n$ are the positive roots of $\tanh \frac{1}{2}\lambda + \tan \frac{1}{2}\lambda = 0$. These functions have been tabulated by Reid & Harris (1958).

linear homogeneous equations for $a_1 \dots a_M$, $b_1 \dots b_M$. A determinantal equation of order $2M$ for $T$ as a function of $R$, $a$, and $\xi$ is obtained from the necessary condition that the determinant of the coefficients vanish. Computations have been carried out for $M = 1$ and 2 and the results are tabulated in table 1. The results for $M = 2$ are also shown graphically in figure 2. It is clear from table 1 that the results even for $M = 1$ are satisfactory, differing by about 1 % or less from the results for $M = 2$, and in close agreement with the results obtained from the exact computations. The success of the Galerkin methods with $M = 1$ for this problem rests essentially on the fact that there are no singular points in equations (7) and (8); hence the characteristic function corresponding to the smallest positive value of $T$ can be easily approximated. Further, for $M = 1$ the computations are particularly easy, requiring only the evaluation of some simple integrals, Finally, it should be pointed out that the solution of the characteristic problem by the Galerkin method as it has just been described is completely equivalent to the substitution of the series given by (14) in the expressions for $\phi$ and $\psi$ and then requiring $I$ to be stationary as a function of the $a_n$ and $b_n$.

When this work was nearly completed the author learned that Chandrasekhar (1960) had considered the characteristic value problem defined by equations (7) and (8) and the boundary conditions (6) with $R$ replaced* by $-R$ and the variables $u$ and $v$ redefined so that the right-hand sides of equations (7) and (8) are $v$ and $-Ta^2u$, respectively. The problem is solved by the methods developed and used by Chandrasekhar in a series of papers. The function $v(x)$ is expanded in a set of complete functions satisfying the boundary conditions $v = 0$ at $x = \pm\frac{1}{2}$, and equation (7) is then solved for $(u)$ the four constants of integration being determined by the boundary conditions $u = Du = 0$ at $x = \pm\frac{1}{2}$. The series for $u$ and $v$ are then substituted in the second-order equation for $v$ and a characteristic determinantal equation is obtained by requiring that the error in the differential equation be orthogonal to the original expansion functions. This method is an improvement on the Galerkin technique but it does require in this case a considerable amount of complex arithmetic. On the other hand, it should be pointed out that the size of the determinantal equation increases like $M$ rather than $2M$ as was the case earlier. The results found by Chandrasekhar (1960) using one term in the series for $v$ are in agreement with the results obtained by the Galerkin method and the exact solution. They are recorded in table 1.

## 4. Case 2; Ω approximated by its average value

In this case (3) and (4) may be rewritten as

$$(D^2-a^2)^2 u - i\{\xi + aR[1-6(\tfrac{1}{4}-x^2)]\}(D^2-a^2)u + 12iaRu = -Ta^2v, \quad (15)$$

$$(D^2-a^2)v - i\{\xi + aR[1-6(\tfrac{1}{4}-x^2)]\}v = u, \quad (16)$$

where again we have let $\xi = \beta - aR$. These equations are identical with (7) and (8) for case 1 except for the bracketed term $1-6(\frac{1}{4}-x^2)$. This term would be

* This simply means that the axial velocity is in the opposite direction to that chosen here, and hence since the disturbance velocities have the same form the values of $\sigma$ ($\beta$ in this paper) tabulated by Chandrasekhar in table 1 of his paper should be prefixed by a minus sign. This was a typographical error.

replaced by zero if we approximate the axial velocity by its average value. Because of the presence of this term the characteristic value problem defined by (15) and (16) and the boundary conditions (6) is considerably more difficult than the characteristic value problem of case 1. Neither the method used earlier for the exact solution nor the method suggested by Chandrasekhar is applicable here.

Approximate values of $T$ can be found using the Galerkin method. Again the solution can be split into even and odd functions. For the even solution we use the series given in (14) for $u$ and $v$ and proceed in the manner described in the

| Investigation | $R$ | $a$ | $\beta$ | $\beta/aR$ | $T$ | $T_1/T_2$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 3·11 | 3·63 | — | 1729 | — |
| 2 | 1 | 3·12 | 3·65 | 1·17 | 1710 | 1·01 |
| 3 | 1 | 3·11 | 2·53 | 0·814 | 1710 | — |
| 1 | 2 | 3·11 | 7·26 | — | 1732 | — |
| 2 | 2 | 3·12 | 7·30 | 1·17 | 1714 | 1·01 |
| 3 | 2 | 3·12 | 5·08 | 0·814 | 1714 | — |
| 1 | 5·17 | 3·11 | 18·77 | — | 1750 | — |
| 2 | 5·17 | 3·12 | 18·9 | 1·17 | 1744 | 1·003 |
| 3 | 5·17 | 3·13 | 13·16 | 0·812 | 1750 | — |
| 1 | 10·34 | 3·15 | 38·05 | — | 1815 | — |
| 2 | 10·34 | 3·13 | 37·8 | 1·17 | 1852 | 0·980 |
| 3 | 10·34 | 3·17 | 26·68 | 0·813 | 1870 | — |
| 1 | 20·67 | 3·28 | 79·40 | — | 2068 | — |
| 2 | 20·67 | 3·15 | 75·8 | 1·16 | 2293 | 0·902 |
| 3 | 20·67 | 3·40 | 57·52 | 0·818 | 2338 | — |
| 1 | 40 | 3·95 | 187·5 | — | 2887 | — |
| 2 | 40 | 3·2 | 147·7 | 1·15 | 4066 | 0·710 |
| 3 | 40 | 4·23 | 142·0 | 0·839 | 3860 | — |
| 1 | 60 | 4·73 | 342·6 | — | 3961 | — |
| 2 | 60 | 3·15 | 215·7 | 1·14 | 7563 | 0·524 |
| 3 | 60 | 5·15 | 267·1 | 0·864 | 5925 | — |

TABLE 2. Critical Taylor numbers and corresponding values of $a$ and $\beta$ and $\beta/aR$ for assigned values of $R$. Investigations 1 and 2 refer to the Galerkin method with $M = 1$, and 2 with a parabolic profile used for $W$. 3 refers to the Galerkin method with $M = 2$ and $W$ approximated by its average value. $T_1/T_2$ is the ratio of the values of $T$ for investigations 1 and 2.

previous section. Actually the only new computations required are the evaluation of the integrals involving $x^2$. Computations have been carried out for $M = 1$ and 2, and the results are tabulated in table 2. The results from case 1 using the Galerkin method with $M = 2$ are also shown in table 2, and the results for cases 1 and 2 with $M = 2$ are given graphically in figure 2.

There are two points that should be made about these results. First, in case 2 as $R$ increases beyond about a value of 20–30 there is an increasing difference between the results of the first and second approximations. This indicates that when the parabolic profile is used for the axial velocity the characteristic functions are more difficult to approximate. Practically speaking, since there is little difference between the first and second approximations up to values of $R \sim 20$, we might expect the second approximation to be satisfactory up to values of

$R \sim 40$. Of course, the only way we can be sure the second approximation is satisfactory is to compute a third approximation and compare the two. This would require the solution of a sixth-order determinantal equation with complex entries, which was not done in the present work. Further it should be noted that the two-term approximation predicts only a very slight increase of critical wave-number with increasing $R$ up to values of about 40–50 and then a slight decrease. However, it would be dangerous to draw any conclusions about this decrease in $a_c$ at $R = 60$ with the information available.

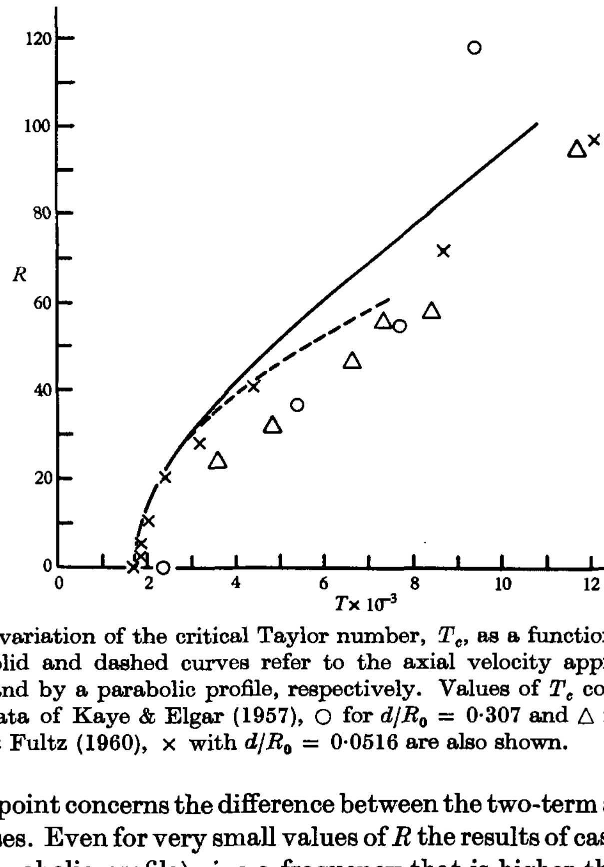


FIGURE 2. The variation of the critical Taylor number, $T_c$, as a function of the Reynolds number, $R$. Solid and dashed curves refer to the axial velocity approximated by its average value and by a parabolic profile, respectively. Values of $T_c$ computed from the experimental data of Kaye & Elgar (1957), ○ for $d/R_0 = 0{\cdot}307$ and △ for $d/R_0 = 0{\cdot}198$; and Donnelly & Fultz (1960), × with $d/R_0 = 0{\cdot}0516$ are also shown.

The second point concerns the difference between the two-term approximations for the two cases. Even for very small values of $R$ the results of case 2 ($W$ approximated by a parabolic profile) give a frequency that is higher than that for the case 1 ($W$ approximated by its average value). As $R$ increases beyond a value of about 20–30 the values of $T_c$ and $a_c$ for the two cases also begin to diverge; $T_c$ for case 2 grows more rapidly than for case 1, and $a_c$ for case 2 remains nearly constant while increasing for case 1. These differences can be traced to the evaluation of the integrals

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \{1-6(\tfrac{1}{4}-x^2)\} C_n(D^2-a^2)\,C_m\,dx \quad \text{and} \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} \{1-6(\tfrac{1}{4}-x^2)\}\,E_n E_m\,dx. \tag{17}$$

If $W(x)$ is approximated by its average value these integrals vanish; on the other hand, when the parabolic profile is used they take on values such that when multiplied by the factor $aR$ they are of the same order of magnitude as the larger contributions from other terms in the differential equations. Indeed for $n = m = 1$ and $a \sim 3$ the first integral in (17) is approximately 10, and when multiplied by $aR$ for $R \sim 20$–60 gives a contribution of about 600–1800. This must be compared with the contributions from the integral of $C_1(D^2-a^2)^2C_1$ from $-\frac{1}{2}$ to $\frac{1}{2}$ which is about 700 for $a \sim 3$.

## 5. Discussion

The results of the present analysis show that when the cylinders are rotating in the same direction the critical Taylor number increases rapidly with increasing Reynolds number. Depending upon whether the axial velocity is approximated by its average values or by a parabolic profile, the results are slightly different. In the latter case, relative to the former the increase in $T_c$ with increasing $R$ is more rapid, the corresponding values of the frequency are greater, and the values of the wave-number are less. The ratio of the wave velocity to the mean average axial velocity is approximately constant up to values of $R \sim 40$ in each case but greater by a factor of about 1·17 in the latter case and less by a factor of about 0·82 in the former case.

In figure 2 the variation of $T_c$ with $R$ for the two cases is shown graphically and also compared with the recent experimental work of Kaye & Elgar (1957) and Donnelly & Fultz* (1960). For values of $R$ up to about 20–30 the results of Donnelly & Fultz are in good agreement with the results for both cases; for larger values of $R$ their measurements tend to confirm the more rapid increase in $T_c$ with increasing $R$ predicted by the use of a parabolic profile for the axial velocity. The results of Kaye & Elgar (1957) are also in qualitative agreement with the theoretical results found here. For small values of $R$ the larger values of $T_c$ that they find may be attributed, in part, to the larger values of $d/R_0$ that were used in their experiments. Indeed for $R = 0$ computation of the critical Taylor number as defined here for the case $k = 0$ and $d/R_0 = \frac{4}{5}$ using the critical value of $\Omega_1$ found by Chandrasekhar (1958) gives a value of $T_c$ of about 3100 in contrast to a value of about 1710 found using a small-gap approximation.

The question of the variation of the wave-number and frequency associated with the secondary motion with $R$ requires more detailed measurements than have been carried out to date. However, Donnelly & Fultz (1960) did make one frequency measurement, finding $\beta \sim 5{\cdot}8$ at $R \sim 2{\cdot}7$. This value of $\beta$ is below the value predicted in both cases 1 and 2. Also, though Kaye & Elgar (1957) did not make any measurements of $\lambda$, Elgar in a letter to the author pointed out that for the range of Reynolds numbers considered here it was his impression that the wavelength changed only slightly with increasing $R$. This would again tend to confirm the results found using a parabolic profile for the axial velocity.

The theoretical results of Goldstein (1937) which predict that $T_c$ will increase with increasing $R$ up to values of about 15–20 and then decrease rapidly with increasing $R$ ($T_c \sim 763$ at $R = 25{\cdot}84$) are not shown in figure 2. As mentioned earlier, the characteristic value problem formulated here is identical with that

* Just as this manuscript was completed Mr Donnelly and Mr Fultz, in a private communication, were kind enough to furnish the author the preliminary data that are shown in figure 2.

treated by Goldstein (1937) Since the results found here in case 1 agree with those found independently and by a different method by Chandrasekhar (1960), it would appear that there must be a numerical error in Goldstein's computations. However, his computations were much too lengthy to check in detail. Also the experimental results of Cornish (1933) and Fage (1938) which are in disagreement with the theoretical results found here and the more recent experimental work are not shown in figure 2. For a comparison of these results with the experimental results of Kaye & Elgar (1957), see the latter paper.

Finally, this analysis shows clearly that the simpler characteristic-value problem with $\Omega$ and $W$ approximated by their average values can be solved easily by several methods. The results of even a one-term approximation using the Galerkin method are satisfactory up to Reynolds number of at least 100. On the other hand, the more physically correct situation with $W$ approximated by a parabolic profile leads to a much more difficult mathematical problem. In this case the use of the Galerkin method with two terms gives results that are acceptable only up to values of the Reynolds number of about 40–50.

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